University of Verona
Ingo Blechschmidt

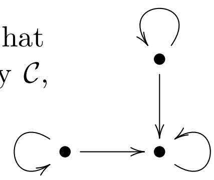
Winter term 2022/2023

Minicourse on Category Theory: Exercise sheet 1

Exercise 1.1. *First steps with categories*

(a) Write down the category for your favorite subject of mathematics or computer science: What are its objects, what are its morphisms? What are its initial and terminal objects?

(b) Write down a formal definition of the category on the right: What precisely are the objects and the morphisms? What is the composition rule?



(c) Prove: Identity morphisms in categories are unique. More precisely, verify that if id_X and $\widetilde{\mathrm{id}}_X$ are both identity morphisms for some object X in a category $\mathcal{C}$, then $\mathrm{id}_X = \widetilde{\mathrm{id}}_X$.

Exercise 1.2. *Initial and terminal objects*

Determine the initial and terminal objects (in case they exist) of ...

(a) the category $\mathrm{Vect}(\mathbb{R})$

(b) the numerical category

(c) the category of Pokémon

(d) the category of Exercise 1.1(b)

Exercise 1.3. *Groups as categories*

Explain how out of any group G (or even just a monoid) a category can be constructed, in such a way that the composition rule of G plays a role.

Remark. The resulting category is sometimes denoted "BG" and arises in algebraic topology.

Exercise 1.4. *Uniqueness up to unique isomorphism*

An *isomorphism* $f : X \to Y$ in a category is a morphism for which a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ exists. We also write "f^{-1}" for g.

(a) Prove: The isomorphisms in the category of sets are precisely the bijective maps.

(b) List two different terminal objects of Set.

(c) Let T and T' be terminal objects of a category $\mathcal{C}$. Show that there is an isomorphism between T and T', and moreover that there is exactly one such isomorphism.

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Minicourse on Category Theory: Exercise sheet 2

Definition. A *monomorphism* in a category $\mathcal{C}$ is a morphism $f : X \to Y$ such that

for all objects Z and all morphisms $p, q : Z \to X$,

if $f \circ p = f \circ q$, then $p = q$.

Dually, an *epimorphism* is a morphism $f : X \to Y$ such that for all objects Z and all morphisms $p, q : Y \to Z$, if $p \circ f = q \circ f$, then $p = q$.

Exercise 2.1. *Monomorphisms and epimorphisms in the category of sets*

Let $f : X \to Y$ be a map between sets. Prove:

(a) The map f is a monomorphism in Set if and only if it is injective.

(b) The map f is a epimorphism in Set if and only if it is surjective.

Remark. This exercise is surprisingly deep from the point of view of mathematical logic: Can you give a direct proof avoiding the law of excluded middle, a proof which could be formalized in IZF? Can you even give a proof avoiding the powerset axiom?

Exercise 2.2. *Composition of monomorphisms*

Let $\mathcal{C}$ be a category.

(a) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. Show: If $g \circ f$ is monic, so is f.

(b) Your proof of (a) works in all categories, hence also in $\mathcal{C}^{\mathrm{op}}$. What does it state then?

Exercise 2.3. *Balanced categories*

(a) Prove: The isomorphisms in the category of monoids are precisely what is commonly known as monoid isomorphisms (bijective monoid homomorphisms).

(b) Prove: Isomorphisms are monic and epic.

Remark. Categories in which the converse holds are called *balanced.* Examples for balanced categories are Set, Grp, Vect($\mathbb{R}$) and many others; however there are also many categories which are not balanced.

(c) Give an example for a category with morphisms which are monic and epic, but fail to be isomorphisms.

Exercise 2.4. *The "2 out of 3" property*

(a) Let f be a morphism in a category which has both a left and a right inverse. Prove that f is an isomorphism and that the given one-sided inverses are equal.

Remark. A *left inverse* to $f : X \to Y$ is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. Dually, a *right inverse* to f is a morphism $h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in a category. Prove that, if two out of the three morphisms f, g and $g \circ f$ are isomorphisms, so is the third.

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Minicourse on Category Theory: Exercise sheet 3

Exercise 3.1. *Product with the terminal object*

Let X be an object of a category $\mathcal{C}$.

(a) Assume that $\mathcal{C}$ contains a terminal object 1. Prove:

$$X \times 1 \cong X.$$

However, this claim is not to be taken literally. Instead, verify more precisely that X (together with which morphisms?) serves as a product of X and 1.

(b) What is the dual statement to (a)?

Exercise 3.2. *Category of wannabe products*

Let X and Y be objects of a category $\mathcal{C}$. We define the following category of *wannabe products* of X and Y:

$$\begin{aligned} \text{objects: } & \text{diagrams of the form } X \leftarrow Q \rightarrow Y \text{ in } \mathcal{C} \\ \text{morphisms: } & \mathrm{Hom}(X \leftarrow Q \rightarrow Y, X \leftarrow R \rightarrow Y) := \end{aligned}$$

$$\left\{ \text{commutative diagrams of the form } \begin{array}{ccccc} & & Q & & \\ & \swarrow & \downarrow & \searrow & \\ X & & & & Y \\ & \nwarrow & \downarrow & \nearrow & \\ & & R & & \end{array} \right\}$$

(a) How should the composition rule be defined in this diagram category?

(b) Explain that a product of X and Y in $\mathcal{C}$ is the same as a terminal object in the category of wannabe products of X and Y.

(c) Which corollary follows from (b) in view of Exercise 1.4?

Exercise 3.3. *Infima in preorders*

Let X be a *preorder*, that is a set X together with a reflexive and transitive (but not necessarily antisymmetric) binary relation $(\preceq)$ on X. For instance $\mathbb{Z}$ with the divisibility ordering is a preorder.

(a) Construct a sensible category BX out of X. Why are the category axioms satisfied?

(b) When are two objects of BX isomorphic?

(c) An *infimum* of two elements $a, b \in X$ is an element $p \in X$ such that

$$\forall x \in X. \qquad x \preceq a \text{ and } x \preceq b \quad \Longleftrightarrow \quad x \preceq p.$$

Show: An infimum of a and b is the same as a product of a and b in BX.

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Minicourse on Category Theory: Exercise sheet 4

Exercise 4.1. *Functors as generalization of group homomorphisms*

(a) Explain how a group homomorphism $\varphi : G \to H$ gives rise to a functor $B(\varphi) : BG \to BH$.

Remark. For a group X, the category BX is the category which has a single object $\heartsuit$ and one morphism $g : \heartsuit \to \heartsuit$ for each group element $g \in X$, with $\mathrm{id}_\heartsuit$ being the neutral element of X and composition given by the operation of the group.

(b) Upgrade your solution of part (a) to a functor $\mathrm{Grp} \to \mathrm{Cat}$.

Exercise 4.2. *Functors between categories induced by preorders*

Let X and Y be preorders. Let BX and BY be the induced thin categories of Exercise 3.3. By which data is a functor $BX \to BY$ given? State and prove an analogue to Exercise 4.1.

Exercise 4.3. *Properties of functors*

(a) Study three functors of your choosing as to whether they are full, faithful or essentially surjective.

(b) Prove that the numerical category is equivalent to its opposite category.

Remark. The numerical category has as objects the natural numbers, and the set of morphisms from m to n is by definition the set of real $(n \times m)$-matrices.

(c) Prove that the functor from the numerical category to the category of finite-dimensional real vector spaces, mapping n to $\mathbb{R}^n$ and a matrix to its inducied linear transformation, is fully faithful and essentially surjective.

Exercise 4.4. *Functoriality of taking products*

Let A be an object of a category $\mathcal{C}$. Assume that we are given, for each object X of $\mathcal{C}$, a product $A \times X$ of A and X. Explain how the incomplete association

$$X \longmapsto A \times X$$

can be extended to a well-defined functor $F : \mathcal{C} \to \mathcal{C}$: How should F be defined on morphisms? Why are the functor axioms satisfied?

Exercise 4.5. *In preparation of the Yoneda lemma I*

Let A be an object of a (locally small) category $\mathcal{C}$. The *contravariant Hom-functor associated to* A is defined by

$$\begin{aligned} \widehat{A} : \quad \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \mathrm{Hom}_{\mathcal{C}}(X, A) \\ (f : X \to Y) &\longmapsto f^\star, \end{aligned}$$

where $f^\star$ is the map

$$\begin{aligned} f^\star : \mathrm{Hom}_{\mathcal{C}}(X, A) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, A) \\ g &\longmapsto g \circ f. \end{aligned}$$

Verify that $\widehat{A}$ is indeed a functor.

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Minicourse on Category Theory: Exercise sheet 5

Exercise 5.1. *Non-equivalent categories*

(a) Let $F : \mathcal{C} \to \mathcal{D}$ be an elementary equivalence of categories (fully faithful, essentially surjective). Show that an object X of $\mathcal{C}$ is initial if and only if $F(X)$ is initial in $\mathcal{D}$.

(b) Find categorical properties which distinguish all of the following categories:

$$\mathrm{Set}, \quad \mathrm{Vect}(\mathbb{R}), \quad \mathrm{Grp}, \quad \mathrm{Pok}, \quad \mathrm{Set}^{\mathrm{op}}.$$

Remark. The category $\mathrm{Set}^{\mathrm{op}}$ obtains a more concrete description by realizing that it is equivalent, by the contravariant powerset functor, to the category of *complete atomic Heyting algebras*. The (only) examples of such algebras are powersets. However this remark is not particularly relevant to the exercise.

Exercise 5.2. *Length as a natural transformation*

Let L be the "underlying functor of the list monad", that is the functor $L : \mathrm{Set} \to \mathrm{Set}$ which maps a set X to the set $L(X)$ of finite lists of elements of X,

$$L(X) = \{[x_1, x_2, \ldots, x_n] \,|\, n \in \mathbb{N}, x_1, \ldots, x_n \in X\},$$

and a map f to the induced map $L(f)$ given by $L(f)([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)]$. For each set X, let $\eta_X : L(X) \to \mathbb{N}$, $\mathsf{xs} \mapsto$ (length of xs).

(a) Show that the maps $(\eta_X)_X$ form a natural transformation $L \Rightarrow K_{\mathbb{N}}$, where $K_{\mathbb{N}}$ is the constant functor on $\mathbb{N}$.

(b) Find a natural transformation $L \circ L \Rightarrow L$ and a natural transformation $\mathrm{Id}_{\mathrm{Set}} \Rightarrow L$.
Remark. Recall that $\mathrm{Id}_{\mathrm{Set}} : \mathrm{Set} \to \mathrm{Set}$ is the identity functor on Set, given by $X \mapsto X$ and $f \mapsto f$.

(c) What would go wrong with finite sets instead of finite lists?

Exercise 5.3. *Examples for natural transformations*

(a) Show: There is just one natural transformation $\eta : \mathrm{Id}_{\mathrm{Set}} \Rightarrow \mathrm{Id}_{\mathrm{Set}}$, namely the family consisting of the maps $\eta_X : X \to X,\ x \mapsto x$.
Hint (also for (b)). Consider suitable maps $\{\heartsuit\} \to X$.

(b) Let $D : \mathrm{Set} \to \mathrm{Set}$ be the functor $X \mapsto X \times X$, $f \mapsto f \times f := ((a, b) \mapsto (f(a), f(b)))$. Show: There is just one natural transformation $\omega : \mathrm{Id}_{\mathrm{Set}} \Rightarrow D$, namely

$$\omega_X : X \to X \times X,\ x \mapsto (x, x).$$

(c) Let $P : \mathrm{Set} \to \mathrm{Set}$ be the (covariant) powerset functor. Show: There is no natural transformation $P \Rightarrow \mathrm{Id}_{\mathrm{Set}}$, but a natural transformation in the other direction.

(d) Assume that we are given, for each inhabited set X, an element $a_X \in X$. Show: The maps $\tau_X : X \to X,\ x \mapsto a_X$ do *not* constitute a natural transformation $\mathrm{Id}_{\mathcal{C}} \Rightarrow \mathrm{Id}_{\mathcal{C}}$, where $\mathcal{C}$ is the category of inhabited sets and arbitrary maps.

(e) Which natural transformations $\mathrm{Id}_{\mathrm{Vect}(\mathbb{R})} \Rightarrow \mathrm{Id}_{\mathrm{Vect}(\mathbb{R})}$ are there?

Exercise 5.4. *In preparation of the Yoneda lemma II*

Let $\varphi : A \to B$ be a morphism in a (locally small) category $\mathcal{C}$. Construct a natural transformation $\widehat{A} \Rightarrow \widehat{B}$ between the Hom functors of Exercise 4.5.

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Minicourse on Category Theory: Exercise sheet 6

Exercise 6.1. *Natural transformations between monotone maps*

Let $f, g : X \to Y$ be monotone maps between preorders X and Y. Let Bf and Bg be the induced functors between the thin categories BX and BY as in Exercise 4.2. Characterize when there is a natural transformation $Bf \Rightarrow Bg$.

Exercise 6.2. *Pseudoinverses of elementary equivalences*

Let $F : \mathcal{C} \to \mathcal{D}$ be an elementary equivalence, so a fully faithful essentially surjective functor. Assume that for each object $Y \in \mathcal{D}$, an object $X_Y \in \mathcal{C}$ with $F(X_Y) \cong Y$ is given. Explain how the mapping $Y \mapsto X_Y$ can be extended to a functor $G : \mathcal{D} \to \mathcal{C}$, and show that $G \circ F \cong \mathrm{Id}_C$ and $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$.

Exercise 6.3. *Linear orderings and permutations*

For a finite set X, let $\mathrm{Perm}(X)$ be the set of bijections from X to X and let $\mathrm{LinOrd}(X)$ be the set of linear orderings on X (lists $[x_1, \ldots, x_n]$ in which every element of X occurs exactly once). Let $\mathcal{S}$ be the category of *species*, that is the subcategory of Set which has as objects the finite sets and as morphisms the bijections.

(a) Show that for every finite set X, $\mathrm{Perm}(X) \cong \mathrm{LinOrd}(X)$.

(b) Explain how Perm and LinOrd can be made into functors $\mathcal{S} \to \mathrm{Set}$.

(c) Show that there is no natural transformation $\mathrm{Perm} \Rightarrow \mathrm{LinOrd}$ and hence also no natural isomorphism $\mathrm{Perm} \cong \mathrm{LinOrd}$.

Hint. Consider the identity permutation.

Exercise 6.4. *Tip already in the diagram*

Let $F : \mathcal{I} \to \mathcal{C}$ be an $\mathcal{I}$-shaped diagram in a category $\mathcal{C}$. Assume that $\mathcal{I}$ contains a terminal object T. Prove that F has a colimit, namely a certain cocone with tip $F(T)$.

Exercise 6.5. *Polynomials and power series*

Let K be a field (or just a commutative ring) and let $K[X]_n$ be the vector space of polynomials of degree $\leq n$ with coefficients in K.

(a) Verify that $K[X]$ can be made into a colimit of the diagram $(K[X]_0 \hookrightarrow K[X]_1 \hookrightarrow K[X]_2 \hookrightarrow \cdots)$, where the morphisms are all the inclusions.

(b) Verify that the vector space $K[\![X]\!] := \{\sum_{i=0}^{\infty} a_i X^i \mid a_0, a_1, \ldots \in K\}$ of formal power series can be made into a limit of the diagram $(\cdots \twoheadrightarrow K[X]_2 \twoheadrightarrow K[X]_1 \twoheadrightarrow K[X]_0)$, where the maps truncate the highest coefficients.

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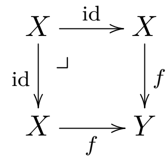
Minicourse on Category Theory: Exercise sheet 7



Exercise 7.1. *Monomorphisms as limits*

(a) Verify that a morphism $f : X \to Y$ in a category is monic if and only if the diagram on the right is a pullback sqaure.

(b) Let $\eta : F \Rightarrow G$ be a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$. Assume that $\mathcal{D}$ has all fiber products. Prove that η is a monomorphism in $[\mathcal{C}, \mathcal{D}]$ if and only if all components η_X are monomorphisms in $\mathcal{D}$.

Hint. Limits in functor categories are computed objectwise.

Exercise 7.2. *The Yoneda lemma*

Let $\mathcal{C}$ be a (small) category and let $\mathrm{PSh}(\mathcal{C}) := [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ be its presheaf category. Our goal is to prove the *Yoneda lemma*, stating that there is a bijection

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\widehat{X}, F) \cong F(X), \tag{$\star$}$$

natural in $X \in \mathrm{Ob}(\mathcal{C})$ and $F \in \mathrm{Ob}(\mathrm{PSh}(\mathcal{C}))$, where $\widehat{X}$ is the functor from Exercise 4.5.

(a) Prove that a natural transformation $\eta : \widehat{X} \Rightarrow F$ is already uniquely determined by the single value $s := \eta_X(\mathrm{id}_X) \in F(X)$, namely by the relationship

$$\eta_Y(f) = F(f)(s) \tag{$\dagger$}$$

for all objects Y and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

(b) Show that conversely, every $s \in F(X)$ defines a natural transf. $\eta : \widehat{X} \Rightarrow F$ by ($\dagger$).

(c) Verify that there is a bijection ($\star$) for fixed $X \in \mathrm{Ob}\,\mathcal{C}$ and $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$.

(d) Domain and codomain of ($\star$) can be regarded as evaluations of the functors

$$\begin{aligned} L &: \mathcal{C}^{\mathrm{op}} \times \mathrm{PSh}(\mathcal{C}) \longrightarrow \mathrm{Set}, \quad (X, F) \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\widehat{X}, F) \\ R &: \mathcal{C}^{\mathrm{op}} \times \mathrm{PSh}(\mathcal{C}) \longrightarrow \mathrm{Set}, \quad (X, F) \longmapsto F(X) \end{aligned}$$

at (X, F). Explain how these functors act on morphisms, and prove that they are isomorphic, concluding your proof of Yoneda's lemma. Congratulations!